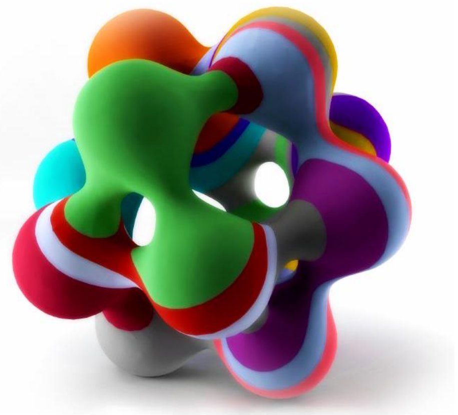

The Euler Characteristic through Morse Theory

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Abstract

This poster introduce the Morse theory and its application in the computation of the Euler characteristic of a manifold. It also gives a intuitive way to understand Poincaré duality by Morse Lemma. This poster assumes the knowledge of basic algebraic topology.

Introduction to Morse Theory

Morse theory is to study the topology of a manifold M by analyzing the critical points of a smooth function $f : M \to \mathbb{R}$. The function f is called a **Morse function** if all of its critical points are non-degenerate. The classical instance is to consider the height function defined on a spere S^2 illustrated in Figure 1, which is a Morse function with two critical points A and B, a maximum and a minimum, and we notice that moving upward along the value of the function, the level sets all have the same topology until we reach the critical points. Another important aspect of Morse theory is the **Morse lemma** which states: Let $f : M \to \mathbb{R}$ be a Morse function and p be a non-degenerate critical point of f. Then there exists a chart (called a **Morse chart**) $(x_1, \ldots, x_n)$ around p such that, on the chart

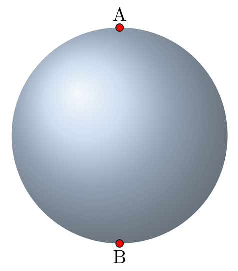

Figure 1: Sphere

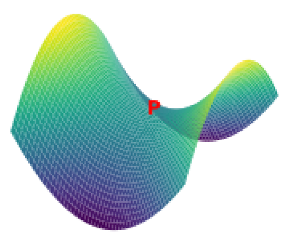

Figure 2: Saddle surface

$$f(x) = f(x_1, \ldots, x_n) = f(p) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2$$

where i is the **index** of p. In 2 or 3 dimensions, the index can be understood intuitively as the number of "linearly independent decreasing directions" at a critical point. For instance, consider a Morse function defined on the saddle surface shown in Figure 2. The index of P is 1, indicating a single linearly independent decreasing direction.

Pseudo-gradient and CW complex

Definition: (Pseudo-gradient): Let $f : M \to \mathbb{R}$ be a Morse function. A pseudo-gradient adapted to f is a **vector field** X on M such that:

- $< \nabla_x f, X_x > \leq 0$($<>$ denotes inner product), where equality holds if and only if x is a critical point of f.
- In a Morse chart around a critical point x, X agrees with $-\nabla f$ for the canonical metric on $\mathbb{R}^n$.

The vector flows of X are called **trajectories** of X flowing from the region of high values towards the region of low values and connecting the critical points.
Definition (CW complex):A CW complex is a topological space built by attaching cells of different dimensions along their boundaries.

Morse complex and Morse Homology

Definition (Morse complex): Let $f : M \to \mathbb{R}$ be a Morse function, $\mathrm{Crit}_k(f)$ denote the set of critical points c_k of f and $n_X(c_{k+1}, c_k)$ be the number of the trajectories of X going from c_{k+1} to c_k. The Morse complex of f is a complex defined as:

$$\cdots \to C_{k+1}(f,R) \xrightarrow{\partial_{k+1}} C_k(f,R) \xrightarrow{\partial_k} C_{k-1}(f,R) \to \cdots$$

where $C_k(f,R) = \left\{\sum_{c\in \mathrm{Crit}_k(f)} a_c c \mid a_c \in R\right\}$ for some ring R and the **boundary map** $\partial_{k+1} : C_{k+1}(f,R) \to \partial_k(f,R)$ as $\partial(c_{k+1}) = \sum_{c\in \mathrm{Crit}_k(f)} n_X(c_{k+1}, c_k)c_k$ where $n_X(c_{k+1}, c_k)$ denotes the number of trajectories of X going from c_{k+1} to c_k. The Morse complex could form a CW complex if the pseudo-gradient X satisfying the **Smale condition**: if all stable and unstable manifolds intersect transversally.
Definition(k-th Morse Homology group): The k-th Morse Homology group is the quotient $H_k(f,R) = \mathrm{Ker}\partial_k/\mathrm{Im}\partial_{k+1}$ and we name $\dim H_k(f,R)$ as **Betti number** $b_k(M)$. For the the height function h defined on the "sphere" in Figure 3, it has 4 critical points, one of index $0(a)$, one of index $1(b)$, and two of index $2(c,d)$. By the definitions above, we find that for the Morse homology of the "sphere" $H_k(h, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for $k \in \{0, n\}$ but 0 otherwise, which is a 2 **mod Morse homology**. For the Klein bottle as depicted in Figure 4, we can attain the **integral Morse homology** of it $H_k(h, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for $k = 1$, $\mathbb{Z}$ for $k = 0$ and 0 otherwise. Furthermore, A **Reeb graph** could be described by Morse function f as the nodes correspond to the critical sets of $f^{-1}(c)$ and edges meet at the nodes reflects the change in topology of the level set $f^{-1}(t)$ as t pass through the critical point c. For instance, the Reeb graph of the height function on a torus is depicted in Figure 5.

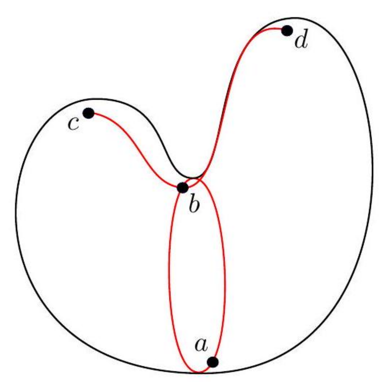

Figure 3: "Sphere"

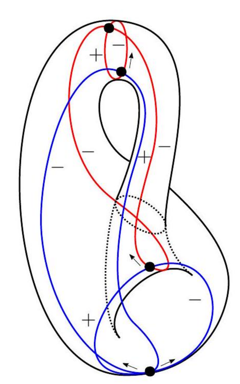

Figure 4: Klein bottle

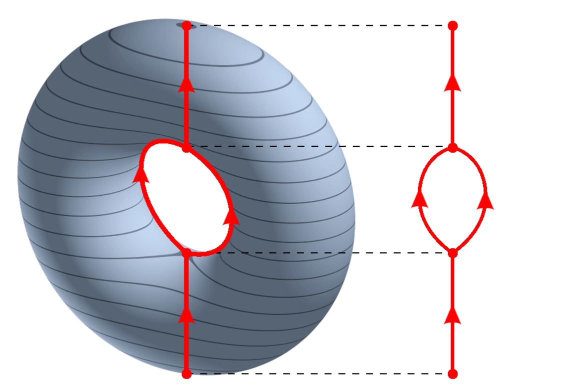

Figure 5: Reeb graph of a torus

Euler Characteristic

The **Euler characteristic** $\chi(M)$ of a manifold M is defined as $\chi(M) = \sum_{k=0}^{n}(-1)^k b_k(M)$. Next, we will prove the following equation only using rank-nullity theorem and basic algebra:

$$\chi(M) = \sum_{k=0}^{n}(-1)^k b_k(M) = \sum_{k=0}^{n}(-1)^k \dim C_k(f)$$

Proof: Let us consider the following Morse complex associated with some manifold M:

$$0 \xrightarrow{\partial_{n+1}} C_n(f) \xrightarrow{\partial_n} C_{n-1}(f) \to \cdots \to C_1(f) \xrightarrow{\partial_1} C_0(f) \xrightarrow{\partial_0} 0$$

$C_k(f)$ are vector spaces connected by the boundary maps ∂_k which is a linear map, then the rank-nullity theorem gives that $\dim\ker\partial_k + \dim \mathrm{Im}\partial_k = \dim C_k(f)$. Hence,

$$\sum_{k=0}^{n}(-1)^k \dim C_k(f) = \sum_{k=0}^{n}(-1)^k[\dim \mathrm{Ker}\partial_k + \dim \mathrm{Im}\partial_k] = \sum_{k=0}^{n}(-1)^k[\dim \mathrm{Ker}\partial_k - \dim \mathrm{Im}\partial_{k+1}] = \sum_{k=0}^{n}(-1)^k b_k(M)$$

It tells us that the homology is independent of the choice of Morse function. Furthermore, if we define $\#\mathrm{Crit}(f)$ to be the total number of the critical points of f, then we have

$$\#\mathrm{Crit}(f) = \sum_{k=0}^{n}[\dim\ker\partial_k + \dim \mathrm{Im}\partial_{k+1}] \geq \sum_{k=0}^{n}[\dim\ker\partial_k - \dim \mathrm{Im}\partial_{k+1}] = \sum_{k=0}^{n} b_k(M)$$

The inequality holds since $\dim \mathrm{Im}\partial_{k+1} \geq 0$. This is called **Morse Inequality**, stating that the number of critical points of a Morse function is at least equal to the sum of the Betti numbers of the manifold.

The Poincaré Duality

By Morse Lemma, the critical points of index k of f are the critical points of index $n-k$ of $-f$. Then, $C_{n-k}(-f)$ is isomorphic to $C_k(f)$ since they have have the same basis. We will get a complex of $C_{n-k+1}(-f)$ and have the following results called the **Poincaré Duality**, which states that for a closed oriented n-manifold M, $b_k(M) = b_{n-k}(M)$. It was first proposed by Henri Poincaré (Figure 6) in 1893. It can be observed that a triangulated manifold provides insight into the existence of a dual polyhedral decomposition. This decomposition is a collection of cells, where each k-cell in the dual polyhedral decomposition corresponds uniquely with an $(n-k)$-cell in the original triangulation. This generalizes the concept of dual polyhedra, as illustrated in Figure 7.

Figure 6: Henri Poincaré

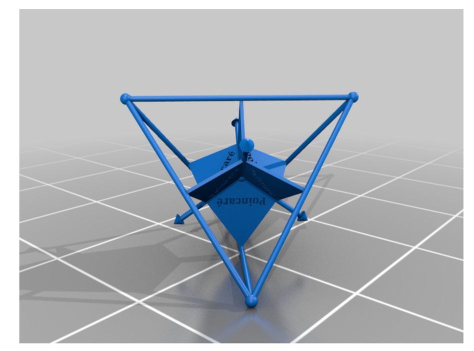

Figure 7: Dual polyhedral decomposition

One Example

Euler characteristic of sphere S^n can calculateed as $\chi(S^n) = 1 + (-1)^n$ by before sections. As an instance, we can consider a 2-D sphere as shown in Figure 8, whose Euler characteristic can be gained by apply the "Euler's polyhedral formula" to a cube as depicted in Figure 9 since they are topological equivalent to each other: $\chi(S^2) = \chi(C) = V - E + F = 8 - 12 + 6 = 2$. This is the same as the result that we calculate by Morse theory. Furthermore, by Poincaré Duality, the Euler characteristic of a compact manifold of odd dimension without boundary is 0 since the terms in the alternating sum cancel each other in pairs.

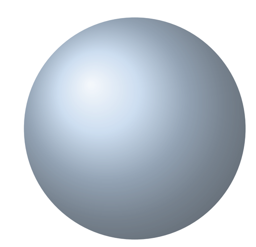

Figure 8: A sphere

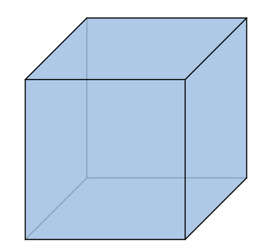

Figure 9: A cube

Acknowledgement and references

The images of the poster are generated by Python, TikZ, and academic sources.

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