Wave equations: Solution by Spherical Means

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July 24, 2023

Contents

1 Preliminaries

1.1 Notations

(i) $Du = (u_{x_1}, u_{x_2}.u_{x_3}, \cdots, u_{x_n})$ is the gradient of u.

(ii) $\Delta =$ Laplacian operator $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$.

(iii) $C^k(U) = \{u : U \to \mathbb{R} \mid u$ is k-times continuously differentiable$\}$.

(iv) $B(\mathbf{x}, r) =$ closed ball with center $\mathbf{x}$ and radius r.

(v) $\partial U =$ boundary of U, $\overline{U} = U \cup \partial U =$ closure of U.

(vi) $\alpha(n) =$ volume of unit ball $B(\mathbf{0}, 1)$ in $\mathbb{R}^n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

(vii) $n\alpha(n) =$ surface area of unit sphere $\partial B(\mathbf{0}, 1)$ in $\mathbb{R}^n$, where Γ is the Gamma function: $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$.

(viii) Averages:

$$\begin{aligned} \fint_{B(\mathbf{x},r)} f dy &= \frac{1}{\alpha(n) r^n} \int_{B(\mathbf{x},r)} f dy \\ &= \text{avarage of } f \text{ over the ball } B(\mathbf{x}, r) \\ \fint_{\partial B(\mathbf{x},r)} f dS &= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(\mathbf{x},r)} f dS \\ &= \text{average of } f \text{ over the sphere } \partial B(\mathbf{x}, r) \end{aligned}$$

(ix) If $\partial U \in C^1$, then along ∂U is defined as the *outward pointing* unit normal vector field:

$$\nu = (\nu_1, \nu_2, \cdots, \nu_n)$$

The unit normal at any point $x^0 \in \partial U$ is $\nu(x^0) = \nu = (\nu_1, \nu_2, \cdots, \nu_n)$.

(x) Let $u \in C^1(\overline{U})$. Then the (outward) *normal derivative* of u along ∂U is defined as:

$$\frac{\partial u}{\partial \nu} := \nu \cdot Du = \sum_{i=1}^{n} \nu_i u_{x_i}$$

1.2 Gauss-Green's Theorem

Suppose $u \in C^1(\overline{U})$ and $U \subset \mathbb{R}^n$ is a bounded domain with smooth boundary ∂U. Then

$$\int_U u_{x_i} dx = \int_{\partial U} u \nu^i dS$$

1.3 Green's Formula

Let $u, v \in C^2(\overline{U})$ and $U \subset \mathbb{R}^n$ be a bounded domain with smooth boundary ∂U. Then

(i) $\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$.

(ii) $\int_U Dv \cdot Du dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} dS - \int_U u \Delta v dx$,

(iii) $\int_U u\Delta v - v\Delta u dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}$.

1.4 Theorem: Polar Coordinates

(i) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continous and summable. Then

$$\int_{\mathbb{R}^n} f dx = \int_0^\infty \left(\int_{\partial B(\mathbf{x}_0, r)} f dS \right) dr$$

for each point $\mathbf{x}_0 \in \mathbb{R}^n$.

(ii) In particular,

$$\frac{d}{dr} \left(\int_{B(\mathbf{x}_0, r)} f d\mathbf{x} \right) = \int_{\partial B(\mathbf{x}_0, r)} f dS$$

for each $r > 0$.

1.5 Transport equation: initial value problem

Consider the following initial value problem:

$$\begin{cases} u_t + b \cdot Du = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

The solution is

$$u(\mathbf{x}, t) = g(x - bt) \quad (\mathbf{x} \in \mathbb{R}^n, t \geq 0) \tag{1}$$

1.6 Transport equation: nonhomogeneous problem

Consider the following nonhomogeneous problem:

$$\begin{cases} u_t + b \cdot Du = f(\mathbf{x}, t) \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{2}$$

The solution is

$$u(\mathbf{x}, t) = g(\mathbf{x} - t\mathbf{b}) + \int_0^t f(\mathbf{x} + (s - t)\mathbf{b}, s) ds \quad (\mathbf{x} \in \mathbb{R}^n, t \geq 0) \tag{3}$$

2 Solution by spherical means

We consider the initial-value problem for the wave equation in n dimensions,

$$\begin{cases} u_{tt} - c^2\Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g, u_t = h & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{4}$$

2.1 Solution for n = 1, d'Alembert's formula

For the one dimensional wave equation in all of $\mathbb{R}$:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t=0\}. \end{cases} \tag{5}$$

where g, h are given functions. We desire to derive a formula for u in terms of g, h. we desire to derive a formula for u in terms of g, h. We use the method of spherical means. We notice that (5) could be "factored" as:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0 \tag{6}$$

We could write

$$v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t) = u_t(x,t) - u_x(x,t) \tag{7}$$

Then, we have

$$\begin{aligned} v_t &= u_{tt} - u_{xt} \\ v_x &= u_{tx} - u_{xx} \end{aligned}$$

We could sum them to get

$$v_t(x,t) + v_x(x,t) = 0 \quad (x \in \mathbb{R}, t > 0) \tag{8}$$

Whereas, (8) is a transport equation with constant coefficients $b = 1$ and $n = 1$. We apply (1) and get

$$v(x,t) = a(x-t) \tag{9}$$

with $a(x) := v(x,0) = u_t(x,0) - u_x(x,0)$. Combining now (7) and (9), we get

$$u_t(x,t) - u_x(x,t) = a(x-t) \quad \text{in} \quad \mathbb{R} \times (0,\infty) \tag{10}$$

We notice that (10) is a nonhomogeneous transport equation, so (3) with $n = 1, \mathbf{b} = -1, f(x,t) = a(x-t)$ implies

$$u(x,t) = b(x+t) + \int_0^t a(x + (t-s) - s)ds = \frac{1}{2}\int_{x-t}^{x+t} a(y)dy + b(x+t) \tag{11}$$

where $b(x) := u(x,0)$. Lastly, we invoke the initial conditions in (5) to compute a and b. For the first intial condition, We set $t = 0$ in (11), then we get

$$u(x,0) = b(x) = g(x) \quad (x \in \mathbb{R}) \tag{12}$$

For the second initial condition, we differentiate (11) with respect to t

$$u_t(x,t) = \frac{1}{2}[a(x+t) + a(x-t)] + b'(x+t) \quad (x \in \mathbb{R}, t \geq 0)$$

Then, let $t = 0$

$$u_t(x,0) = \frac{1}{2}[a(x)+a(x)]+b'(x) = a(x)+b'(x) = h(x) \implies a(x) = h(x)-b'(x) \quad (x \in \mathbb{R})$$

By (12),

$$a(x) = h(x) - g'(x) \quad (x \in \mathbb{R})$$

We subtitude this into (11)

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t}[h(y) - g'(y)]dy + g(x+t) \quad (x \in \mathbb{R}, t \geq 0)$$

Then using the fundamental theorem of calculus, we get

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(y)dy + \frac{1}{2}[g(x+t) + g(x-t)] \quad (x \in \mathbb{R}, t \geq 0) \tag{13}$$

This is *d'Alembert's formula* for the one dimensional wave equation. We could assume u is sufficiently smooth and check that this is really a solution of (5).

Theorem 2.1. *(Solution of wave equation, $n = 1$) Assume $g \in C^2(\mathbb{R}), h \in C^1(\mathbb{R})$ and define u by d'Alembert's formula* (13)*.*

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} h(y)dy + \frac{1}{2}[g(x+t) + g(x-t)] \quad (x \in \mathbb{R}, t \geq 0)$$

Then,

(i) $u \in C^2(\mathbb{R} \times [0,\infty))$,

(ii) $u_{tt} - u_{xx} = 0 \quad$ *in* $\mathbb{R} \times (0,\infty)$,

(iii) $\lim_{(x,t)\to(x^0,0^+)} u(x,t) = g(x^0)$, $\lim_{(x,t)\to(x^0,0^+)} u_t(x,t) = h(x^0)$ *for each point* $x^0 \in \mathbb{R}$.

Proof. We assume $g \in C^2(\mathbb{R}), h \in C^1(\mathbb{R})$ and define u by d'Alembert's formula (13). Then, u is clearly C^2 in $\mathbb{R} \times (0,\infty)$. We now check that u satisfies the wave equation. We differentiate u with respect to t and x.

$$\begin{aligned} u_t(x,t) =& \frac{1}{2}[h(x+t) + h(x-t)] + \frac{1}{2}[g'(x+t) - g'(x-t)] \\ u_x(x,t) =& \frac{1}{2}[h(x+t) - h(x-t)] + \frac{1}{2}[g'(x+t) + g'(x-t)] \end{aligned}$$

Then, we compute u_{tt} and u_{xx}.

$$\begin{aligned}u_{tt}(x,t) =& \frac{1}{2}[h'(x+t) - h'(x-t)] + \frac{1}{2}[g''(x+t) + g''(x-t)]\\ u_{xx}(x,t) =& \frac{1}{2}[h'(x+t) + h'(x-t)] + \frac{1}{2}[g''(x+t) + g''(x-t)]\end{aligned}$$

It is clear that $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times (0,\infty)$. Lastly, we check the initial conditions. We first check that $\lim_{(x,t)\to(x^0,0)} u(x,t) = g(x^0)$ for each $x^0 \in \mathbb{R}$. We fix $x^0 \in \mathbb{R}$ and let $(x,t) \to (x^0, 0)$. Then, $x - t \to x^0$ and $x + t \to x^0$. Thus, by continuity of g,

$$\lim_{(x,t)\to(x^0,0^+)} u(x,t) = \frac{1}{2}\int_{x^0}^{x^0} h(y)dy + \frac{1}{2}[g(x^0) + g(x^0)] = g(x^0)$$

Next, we check that $\lim_{(x,t)\to(x^0,0)} u_t(x,t) = h(x^0)$ for each $x^0 \in \mathbb{R}$. We fix $x^0 \in \mathbb{R}$ and let $(x,t) \to (x^0, 0)$. Then, $x - t \to x^0$ and $x + t \to x^0$. Thus, by continuity of h,

$$\lim_{(x,t)\to(x^0,0^+)} u_t(x,t) = \frac{1}{2}[h(x^0) + h(x^0)] + \frac{1}{2}[g'(x^0) - g'(x^0)] = h(x^0)$$

$\square$

Remark 2.1. *(i) Observing* (13)*, we see that the solution u has the form*

$$u(x,t) = F(x+t) + G(x-t)$$

for some function F and G. Conversely, and function of this form solves the wave equation $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times (0,\infty)$. Also, $F(x+t)$ is the general solution of $u_t - u_x = 0$ and $G(x-t)$ is the general solution of $u_t + u_x = 0$. Hence, the general solution of the one-dimentional wave equation is the sum of the general solution of $u_t - u_x = 0$ and the general solution of $u_t + u_x = 0$. This is the consequence of the factorization in (6).

(ii) We see that from 13, if $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k(\mathbb{R} \times [0,\infty))$ but is not general smoother. Thus, the wave equation does not couse instantaneous smoothing of the initial data.

2.1.1 A reflection method

To illustrate a further application of d'Alembert's formula, we consider this initial/boundary value problem for the wave equation on the half-line $\mathbb{R}_+ = (0,\infty)$:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u(x,0) = g(x), u_t(x,0) = h(x) & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u(0,t) = 0 & \text{on } \{x = 0\} \times (0,\infty) \end{cases} \tag{14}$$

where g, h are given with $g(0) = h(0) = 0$. We can solve this problem by extending u, g, h to the whole line $\mathbb{R}$ by odd reflection. That is, we define

$$\widetilde{u}(x,t) := \begin{cases} u(x,t) & (x \geq 0, t \geq 0) \\ -u(-x,t) & (x \leq 0, t \geq 0) \end{cases}$$

$$\widetilde{g}(x) := \begin{cases} g(x) & (x \geq 0) \\ -g(-x) & (x \leq 0) \end{cases}$$

$$\widetilde{h}(x) := \begin{cases} h(x) & (x \geq 0) \\ -h(-x) & (x \leq 0) \end{cases}$$

We could differentiate $\widetilde{u}$ with respect to x and t and obtain

$$\widetilde{u}_x(x,t) := \begin{cases} u_x(x,t) & (x \geq 0, t \geq 0) \\ u_x(-x,t) & (x \leq 0, t \geq 0) \end{cases}$$

$$\widetilde{u}_t(x,t) := \begin{cases} u_t(x,t) & (x \geq 0, t \geq 0) \\ -u_t(-x,t) & (x \leq 0, t \geq 0) \end{cases}$$

Differentiating $\widetilde{u}_x$ with respect to x and $\widetilde{u}_t$ with respect to t gives

$$\widetilde{u}_{xx}(x,t) := \begin{cases} u_{xx}(x,t) & (x \geq 0, t \geq 0) \\ -u_{xx}(-x,t) & (x \leq 0, t \geq 0) \end{cases}$$

$$\widetilde{u}_{tt}(x,t) := \begin{cases} u_{tt}(x,t) & (x \geq 0, t \geq 0) \\ -u_{tt}(-x,t) & (x \leq 0, t \geq 0) \end{cases}$$

Then, by 14, we have

$$\begin{cases} \widetilde{u}_{tt} - \widetilde{u}_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ \widetilde{u}(x,0) = \widetilde{g}(x), \widetilde{u}_t(x,0) = \widetilde{h}(x) & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

This is the wave equation on the whole line $\mathbb{R}$ with initial data $\widetilde{g}, \widetilde{h}$. By d'Alembert's formula, the solution is

$$\widetilde{u}(x,t) = \frac{1}{2}[\widetilde{g}(x+t) + \widetilde{g}(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \widetilde{h}(y)dy$$

Recalling the definitions of $\widetilde{u}, \widetilde{g}, \widetilde{h}$ above, since $x \geq 0$, then $x + t \geq 0$ but it is uncertain whether $x - t \geq 0$. If $x - t \geq 0$, then $\widetilde{g}(x-t) = g(x-t)$. If $x - t < 0$, then $\widetilde{g}(x-t) = -g(t-x)$. Thus, the solution $\widetilde{u}$ can be written as for $x \geq 0$ and $t \geq 0$:

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x+t) - g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy & \text{if } x \geq t \geq 0 \\ \frac{1}{2}[g(x+t) - g(t-x)] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)dy & \text{if } 0 \leq x \leq t \end{cases} \tag{15}$$

Note that this solution does not belong to C^2, unless $g''(0) = 0$.

2.2 Spherical means

When $c = 1$, we suppose $n \geq 2$, $m \geq 2$, and $u \in C^m(\mathbb{R}^n \times [0, \infty))$ solves this initial evalue problem:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(\mathbf{x}, 0) = g(\mathbf{x}), u_t(\mathbf{x}, 0) = h(\mathbf{x}) & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{16}$$

We try to find an explicit formula of u in terms of g and h. We consider the avarage of u over certain spheres. These avarages are called spherical means as functions of the time t and the radius r. It turns out that to solve the Euler-Poisson-Darboux equation, which is a PDE we can for odd n to convert it into an ordinary one-dimensional wave equation. Thus, we can apply d'Alembert's formula leading a formula for the solution. We introduce some notations firstly:

(i) Let $\mathbf{x} \in \mathbb{R}^n, r > 0$. The **ball average** of f at $\mathbf{x}$ and radius r is defined as:

$$U(\mathbf{x}; r, t) := \fint_{\partial B(\mathbf{x}, r)} u(\mathbf{y}, t) dS(\mathbf{y}) \tag{17}$$

the average of $u(\mathbf{y}, t)$ over the sphere $\partial B(\mathbf{x}, r)$.

(ii) Similarity, for intial condition g and h, we define

$$\begin{cases} G(\mathbf{x}, r) := & \fint_{\partial B(\mathbf{x}, r)} g(\mathbf{y}) dS(\mathbf{y}) \\ H(\mathbf{x}, r) := & \fint_{\partial B(\mathbf{x}, r)} h(\mathbf{y}) dS(\mathbf{y}) \end{cases}$$

Then, we have the following lemma:

Lemma 2.1. *(Euler-Poisson-Darbous equation). Fix $x \in \mathbb{R}^n$, and let u satisfy 16. Then $U \in C^m\left(\overline{\mathbb{R}}_+ \times [0, \infty)\right)$ and*

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \textit{in } \mathbb{R}_+ \times (0, \infty) \\ U(r, 0) = G(r), U_t(r, 0) = H(r) & \textit{on } \mathbb{R}_+ \times \{t = 0\} \end{cases} \tag{18}$$

The partial differential equation (18) is called the Euler-Poisson-Darboux equation. (Note that the term $U_{rr} + \frac{n-1}{r} U_r$ is the ratial part of the Laplacian Δ in polar coordinates.) We prove this lemma as follows:

Proof. 1. We first prove that $U \in C^m\left(\overline{\mathbb{R}}_+ \times [0, \infty)\right)$. We fix $t \geq 0$ and $r \geq 0$. Let $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. We write $U(\mathbf{x}; r, t)$ as:

$$U(\mathbf{x}; t, r) = \fint_{\partial B(\mathbf{x}, r)} u(\mathbf{y}, t) dS(\mathbf{y}) \overset{\mathbf{y} = \mathbf{x} + r\mathbf{z}}{=} \fint_{\partial B(\mathbf{0}, 1)} u(\mathbf{x} + r\mathbf{z}, t) dS(\mathbf{z})$$

We differentiate this with respect to r:

$$U_r = \fint_{\partial B(\mathbf{0}, 1)} Du(\mathbf{x} + r\mathbf{z}, t) \cdot \mathbf{z} dS(\mathbf{z}) \overset{\mathbf{z} = \frac{\mathbf{y} - \mathbf{x}}{r}}{=} \fint_{\partial B(\mathbf{x}, r)} Du(\mathbf{y}, t) \cdot \frac{\mathbf{y} - \mathbf{x}}{r} dS(\mathbf{y})$$

Consequently, by Green's formula, we have:

$$\begin{aligned} U_r(\mathbf{x}; r, t) &= \fint_{\partial B(\mathbf{x},r)} Du(\mathbf{y}, t) \cdot \frac{\mathbf{y} - \mathbf{x}}{r} dS(\mathbf{y}) \\ &= \fint_{\partial B(\mathbf{x},r)} \frac{\partial u(\mathbf{y}, t)}{\partial \nu} dS(\mathbf{y}) \\ &= \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u(\mathbf{y}, t) d\mathbf{y} \end{aligned}$$

From this equality, we deduce that $\lim_{r \to 0^+} U_r(\mathbf{x}; r, t) = 0$. We then differentiate U_r with respect to r again, we use some trick to do this: By the definition of avarage of u over the sphere, we have:

$$r^{n-1} U_r(\mathbf{x}; r, t) = \frac{1}{n\alpha(n)} \int_{B(\mathbf{x},r)} \Delta u(\mathbf{y}, t) d\mathbf{y}$$

We differentiate both sides with respect to r:

$$r^{n-1} U_{rr}(\mathbf{x}; r, t) + (n-1) r^{n-2} U_r(\mathbf{x}; r, t) = \frac{1}{n\alpha(n)} \int_{\partial B(\mathbf{x},r)} \Delta u(\mathbf{y}, t) dS(\mathbf{y})$$

Then, we have the following equality:

$$U_{rr}(\mathbf{x}; r, t) = \fint_{\partial B(\mathbf{x},r)} \Delta u(\mathbf{y}, t) dS(\mathbf{y}) + \left(\frac{1}{n} - 1 \right) \fint_{B(\mathbf{x},r)} \Delta u(\mathbf{y}, t) d\mathbf{y} \quad (19)$$

Therefore, $\lim_{r \to 0^+} U_{rr}(\mathbf{x}; r, t) = \frac{1}{n} \Delta u(\mathbf{x}, t)$. We use (19), we could compute $U_{rrr}(x; r, t)$, etc. Therefore, we could verify that $U \in C^m \left(\overline{\mathbb{R}}_+ \times [0, \infty) \right)$.

2. By the equation in (16), we have:

$$U_r = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u(\mathbf{y}, t) d\mathbf{y} = \frac{r}{n} \fint_{B(\mathbf{x},r)} u_{tt} d\mathbf{y} = \frac{1}{n\alpha(n)} \frac{1}{r^{n-1}} \int_{B(\mathbf{x},r)} u_{tt} d\mathbf{y}$$

Thus, we have:

$$r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(\mathbf{x},r)} u_{tt} d\mathbf{y}$$

We differentiate both sides with respect to r:

$$\begin{aligned} (n-1) r^{n-2} U_r + r^{n-1} U_{rr} &= \frac{1}{n\alpha(n)} \int_{\partial B(\mathbf{x},r)} u_{tt} dS(\mathbf{y}) \\ &= r^{n-1} \fint_{\partial B(\mathbf{x},r)} u_{tt} dS(\mathbf{y}) = r^{n-1} U_{tt} \end{aligned}$$

Then, we could substitute this into (16), we have:

$$U_{tt} = U_{rr} + \frac{n-1}{r} U_r \Rightarrow U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \quad (20)$$

$\square$

2.3 Solution for $\mathbf{n = 3, 2}$, Kirchhoff's and Poisson's formulas

2.3.1 Solution for $\mathbf{n = 3}$

We now consider the case $n = 3$. Therefore, the equation (16) becomes:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{21}$$

The plan is to transfer the Euler-Poisson-Darbous equation into the usual one-dimensional wave equation. We first consider the case $n = 3$. We suppose that $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of the initial value problem (16). We set:

$$\tilde{U} := rU \quad \tilde{G} := rG \quad \text{and} \quad \tilde{H} := rH \tag{22}$$

We now verify that $\tilde{U}$ solves the following initial value problem:

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \tilde{U} = G, \tilde{U}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ \tilde{U} = 0 & \text{on } \{r = 0\} \times (0, \infty) \end{cases} \tag{23}$$

Indeed, we have

$$\begin{aligned} \tilde{U}_{tt} &= rU_{tt} \\ &= r[U_{rr} + \frac{2}{r}U_r] \quad \text{by (20), with } n = 3 \\ &= 2U_{rr} + 2U_r \\ &= (U + rU_r)_r \\ &= \tilde{U}_{rr} \quad \text{by (22)} \end{aligned}$$

It is easy to verify that $\tilde{G}_{rr}(0) = 0$. Therefore, we could apply (15) to (23), for $0 \leq r \leq t$, we have:

$$\tilde{U}(\mathbf{x}; r, t) = \frac{1}{2}\left[\tilde{G}(t + r) + \tilde{G}(t - r)\right] + \int_{t-r}^{t+r} \tilde{H}(y)dy \tag{24}$$

By the definition of the average ball and surface, we have:

$$u(\mathbf{x}, t) = \lim_{r \to 0^+} U(\mathbf{x}; r, t).$$

Therefore, we could conclude that from (22) and (24):

$$
\begin{aligned}
u(\mathbf{x},t) &= \lim_{r\to 0^+} \frac{\tilde{U}(\mathbf{x};r,t)r}{r} \\
&= \lim_{r\to 0^+} \frac{\tilde{U}(\mathbf{x};r,t)r}{r} \\
&= \lim_{r\to 0^+} \frac{1}{2r}\left[\tilde{G}(t+r)+\tilde{G}(t-r)\right] + \int_{t-r}^{t+r} \tilde{H}(y)dy \\
&= \lim_{r\to 0^+} \left[\frac{\tilde{G}(t+r)-\tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(ydy)\right] \\
&= \tilde{G}'(t) + \tilde{H}(t)
\end{aligned}
$$

Now

$$\tilde{G}(\mathbf{x};r) = rG(\mathbf{x};r) = r\fint_{\partial B(\mathbf{x},r)} g(\mathbf{y})dS(\mathbf{y})$$

implies,

$$\tilde{G}(\mathbf{x};t) = tG(\mathbf{x};t) = t\fint_{\partial B(\mathbf{x},t)} g(\mathbf{y})dS(\mathbf{y})$$

Similarly,

$$\tilde{H}(\mathbf{x};t) = rH(\mathbf{x};t) = t\fint_{\partial B(\mathbf{x},t)} h(\mathbf{y})dS(\mathbf{y})$$

Therefore, the solution of wave eqaution in $\mathbb{R}^3$ is given by:

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left(t\fint_{\partial B(\mathbf{x},t)} g(\mathbf{y})dS(\mathbf{y})\right) + t\fint_{\partial B(\mathbf{x},t)} h(\mathbf{y})dS(\mathbf{y}) \tag{25}$$

If g is smooth, then the solution could simplifed further. In particular, for g is enough, we have:

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(t\fint_{\partial B(\mathbf{x},t)} g(\mathbf{y})dS(\mathbf{y})\right) &= \frac{\partial}{\partial t}\left(t\fint_{\partial B(\mathbf{0},1)} g(\mathbf{x}+t\mathbf{z})dS(\mathbf{z})\right) \\
&= \fint_{\partial B(\mathbf{0},1)} g(\mathbf{x}+t\mathbf{z})dS(\mathbf{z}) + t\fint_{\partial B(\mathbf{0},1)} Dg(\mathbf{x}+t\mathbf{z})\cdot\mathbf{z}dS(\mathbf{z}) \\
&= \fint_{\partial B(\mathbf{x},t)} g(\mathbf{y})dS(\mathbf{y}) + t\fint_{\partial B(\mathbf{x},t)} Dg(\mathbf{y})\cdot\left(\frac{\mathbf{y}-\mathbf{x}}{t}\right)dS(\mathbf{y}) \\
&= \fint_{\partial B(\mathbf{x},t)} g(\mathbf{y})dS(\mathbf{y}) + \fint_{\partial B(\mathbf{x},t)} Dg(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x})dS(\mathbf{y})
\end{aligned}
$$

And

$$\tilde{H}(\mathbf{x};t) = tH(\mathbf{x};t) = t\fint_{\partial B(\mathbf{x},t)} h(\mathbf{y})dS(\mathbf{y})$$

Therefore, substitute these into (25), we have:

$$u(\mathbf{x},t) = \fint_{\partial B(\mathbf{x},t)} [th(\mathbf{y}) + g(\mathbf{y}) + Dg(\mathbf{y}) \cdot (\mathbf{y}-\mathbf{x})] dS(\mathbf{y}) \tag{26}$$

Further, we note that in $\mathbb{R}^3$,

$$u(\mathbf{x},t) = \frac{1}{4\pi t^2} \fint_{\partial B(\mathbf{x},t)} [th(\mathbf{y}) + tg(\mathbf{y}) + tDg(\mathbf{y}) \cdot (\mathbf{y}-\mathbf{x})] dS(\mathbf{y}) \tag{27}$$

This is know as the *Kirchhoff's formula* for the solution for the initial value problem of the wave equation in $\mathbb{R}^3$.

Remark 2.2. *Above we found the solution for the wave equation in $\mathbb{R}^3$ in the case where $c = 1$. In fact, when $c \neq 1$, we could use the change of variable to apply the formula above. In particular, consider the initial value problem:*

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & \textit{in } \mathbb{R}^3 \times (0,\infty) \\ u(\mathbf{x},0) = g(\mathbf{x}), & \textit{in } \mathbb{R}^3 \\ u_t(\mathbf{x},0) = h(\mathbf{x}), & \textit{in } \mathbb{R}^3 \end{cases} \tag{28}$$

We suppose that v is a solution of (28). Then, we define $u(\mathbf{x},t) = v(\mathbf{x}, \frac{1}{c}t)$. Then,

$$u_{tt} - \Delta u = \frac{1}{c^2} v_{tt} - \Delta v = 0$$

It implies that u is a solution of

$$\begin{cases} u_{tt} - \Delta u = 0 & x \in \mathbb{R}^3 \times (0,\infty) \\ u(\mathbf{x},0) = g(\mathbf{x}) \\ u_t(\mathbf{x},0) = \frac{1}{c} h(\mathbf{x}) \end{cases}$$

Therefore, u is given by the Kirchhoff's formula. Now, by making the change of variables ot $= \frac{1}{c}t$, we see that

$$v(\mathbf{x},t) = u(\mathbf{x},ct) = \frac{1}{4\pi c^2 t^2} \fint_{\partial B(\mathbf{x},ct)} [th(\mathbf{y}) + g(\mathbf{y}) + Dg(\mathbf{y}) \cdot (\mathbf{y}-\mathbf{x})] dS(\mathbf{y})$$

2.3.2 Solution for $\mathbf{n = 2}$

There is no transformation like (22) working to convert the Euler-Poisson-Darboux equation into one-demensional wave equation when $n = 2$. Instead, we take the initial value problem for $n = 2$:

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u(\mathbf{x},0) = g(\mathbf{x}), & \text{in } \mathbb{R}^2 \\ u_t(\mathbf{x},0) = h(\mathbf{x}), & \text{in } \mathbb{R}^2 \end{cases} \tag{29}$$

and simply regard it as a problem for $n = 3$, in which the third spartial variable is set to be zero. Suppose $u \in C^2(\mathbb{R}^2 \times [0,\infty)$ is a solution of (29). We define

$$\overline{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t) \tag{30}$$

Then, (16) implies that $\overline{u}$ is a solution of

$$\begin{cases} \overline{u}_{tt} - \Delta \overline{u} = 0, & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \overline{u} = \overline{g}, \overline{u}_t = \overline{h}, & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases} \tag{31}$$

for

$$\overline{g}(x_1, x_2, x_3) := g(x_1, x_2),\ \overline{h}(x_1, x_2, x_3) := h(x_1, x_2)$$

If we write $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\overline{\mathbf{x}} = (x_1, x_2, 0) \in \mathbb{R}^3$, then (31) and Kirchoff's formula (25) imply that

$$u(\mathbf{x}, t) = \overline{u}(\overline{\mathbf{x}}, t) = \frac{\partial}{\partial t}\left(t \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{g} d\overline{S}\right) + t \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{h} d\overline{S} \tag{32}$$

where $\overline{B}(\overline{\mathbf{x}}, t)$ is the ball in $\mathbb{R}^3$ centered at $\overline{\mathbf{x}}$ with radius $t > 0$, and $d\overline{S}$ denotes two-dimensional surface measure on $\partial\overline{B}(\overline{\mathbf{x}}, t)$. We can rewrite (32) by observing that

$$\fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{g} d\overline{S} = \frac{1}{4\pi t^2} \int_{\partial \overline{B}(\overline{\mathbf{x}},t)} g(\mathbf{y}) dS(\mathbf{y}) = \frac{2}{4\pi t^2} \int_{B(\mathbf{x},t)} g(\mathbf{y})(1 + |D\gamma(\mathbf{y})|^2)^{1/2} d\mathbf{y}$$

where $\gamma(\mathbf{y}) = (t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}$ for $\mathbf{y} \in B(\mathbf{x}, t)$. There is a "2" in the denominator since $\partial\overline{B}(\overline{\mathbf{x}}, t)$ is the union of two hemispheres. Since $\gamma(\mathbf{y}) = (t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}$ for $\mathbf{y} \in B(\mathbf{x}, t)$, we have

$$D\gamma(\mathbf{y}) = -\frac{\mathbf{y} - \mathbf{x}}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}}$$

which impies that

$$(1 + |D\gamma(\mathbf{y})|^2)^{1/2} = \frac{t}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}}$$

We substitute this into the above equation and obtain

$$\begin{aligned} \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{g} d\overline{S} &= \frac{1}{2\pi t} \int_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y} \\ &= \frac{\alpha(2) t^2}{2\pi t} \fint_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y} \\ &= \frac{t}{2} \fint_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y} \end{aligned}$$

Similarly,

$$\fint_{\partial\overline{B}(\overline{\mathbf{x}},t)} \overline{h} d\overline{S} = \frac{t}{2}\fint_{B(\mathbf{x},t)} \frac{h(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y}$$

Consequently, (32) becomes

$$u(\mathbf{x},t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2\fint_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y}\right) + \frac{t^2}{2}\fint_{B(\mathbf{x},t)} \frac{h(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y}$$

Since

$$t^2\fint_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y} \overset{\mathbf{y}=\mathbf{x}+t\mathbf{z}}{=} t\fint_{B(\mathbf{0},1)} \frac{g(\mathbf{x}+t\mathbf{z})}{(1-|\mathbf{z}|^2)^{\frac{1}{2}}} d\mathbf{z}$$

so

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(t^2\fint_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}}\right) \\
&= \frac{\partial}{\partial t}\left(t\fint_{B(\mathbf{0},1)} \frac{g(\mathbf{x}+t\mathbf{z})}{(1-|\mathbf{z}|^2)^{\frac{1}{2}}} d\mathbf{z}\right) \\
&= \fint_{B(\mathbf{0},1)} \frac{g(\mathbf{x}+t\mathbf{z})}{(1-|\mathbf{z}|^2)^{\frac{1}{2}}} d\mathbf{z} + t\fint_{B(\mathbf{0},1)} \frac{Dg(\mathbf{x}+t\mathbf{z})\cdot\mathbf{z}}{(1-|\mathbf{z}|^2)^{\frac{1}{2}}} d\mathbf{z} \\
&= \fint_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{z} + t\fint_{B(\mathbf{x},t)} \frac{Dg(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{z}
\end{aligned}$$

Therefore, we could rewrite the solution as

$$u(\mathbf{x},t) = \frac{1}{2}\fint_{B(\mathbf{x},t)} \frac{tg(\mathbf{y}) + t^2h(\mathbf{y}) + tDg(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y} \tag{33}$$

for $\mathbf{x} \in \mathbb{R}^2, t > 0$. This is the *Poisson formula* for the solution of the inital value problem (16) in two dimensions. Again, by making a change of variables, we could see that the solution of the wave equaiton in two dimensions is given by

$$u(\mathbf{x},t) = \frac{1}{2c^2}\fint_{B(\mathbf{x},t)} \frac{ctg(\mathbf{y}) + ct^2h(\mathbf{y}) + ctDg(\mathbf{y})\cdot(\mathbf{y}-\mathbf{x})}{(c^2t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y}$$

This trick of solving the problem for $n = 3$ first and then dropping to $n = 2$ is called *method of descent*. It is generally used to find the solution of the wave equation in even dimensions, using the solution of the wave equation in the next higher odd dimensions.

2.3.3 Solution for odd n

Assume now

$$n \text{ is an odd integer, } n \geq 3.$$

We first record some identities that will be useful in the following discussion.

Lemma 2.2. *Let $\phi : \mathbb{R} \to \mathbb{R} \in C^{k+1}$. Then, for $k = 1, 2, \cdots$:*

(i) $\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}\phi(r)) = \left(\frac{1}{r}\frac{d}{dr}\right)^k \left(r^{2k}\frac{d\phi}{dr}(r)\right)$,

(ii) $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}\phi(r)) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}$, *where the constant* $\beta_j^k (j = 0, 1, \ldots, k-1)$ *are independent of* ϕ.

(iii) $\beta_0^k = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)$.

Proof. We prove these by induction:

(i) For $k = 1$, we have

$$\begin{aligned}
\frac{d}{d^2r}(r\phi(r)) &= \frac{d}{dr}\left(\frac{d}{dr}(r\phi(r))\right) \\
&= \frac{d}{dr}\left(\phi(r) + r\frac{d\phi}{dr}(r)\right) \\
&= 2\frac{d\phi}{dr}(r) + r\frac{d^2\phi}{dr^2}(r) \\
&= \frac{1}{r}\left(2r\frac{d\phi}{dr}(r) + r^2\frac{d^2\phi}{dr}(r)\right) \\
&= \frac{1}{r}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}(r)\right)
\end{aligned}$$

Now, assume that the result holds for $k-1$,

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-2}(r^{2k-3}\phi(r)) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-2}\frac{d\phi}{dr}(r)\right)$$

then for k, we have

$$\begin{aligned}
LHS &= \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}\phi(r)) = \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-2}\left((2k-1)r^{2k-3}\phi(r) + r^{2k-2}\frac{d\phi}{dr}(r)\right) \\
RHS &= \left(\frac{1}{r}\frac{d}{dr}\right)^k\left(r^{2k}\frac{d\phi}{dr}(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left((2k)r^{2k-2}\frac{d\phi}{dr}(r) + r^{2k-1}\frac{d^2\phi}{dr^2}(r)\right)
\end{aligned}$$

By the induction hypothesis, the first term of RHS could be merged with the first term of LHS. Therefore, we have

$$
\begin{aligned}
& LHS - RHS \\
&= -\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-2}\left(r^{2k-3}\left(\phi(r) - r\frac{d\phi}{dr}(r)\right)\right) - \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\frac{d^2\phi}{dr^2}(r)\right) \\
&= \frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-2}\left(r^{2k-3}\left(r\frac{d\phi}{dr}(r) - \phi(r)\right)\right) - \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\frac{d^2\phi}{dr^2}(r)\right)
\end{aligned}
$$

Use the induction hypothesis again with $\left(r\frac{d\phi}{dr}(r) - \phi\right)$ to replace ϕ, we have

$$
\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-2}\left(r^{2k-3}\left(r\frac{d\phi}{dr}(r) - \phi(r)\right)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-2}\frac{d}{dr}\left(r\frac{d\phi}{dr}(r) - \phi(r)\right)\right)
$$

Therefore,

$$
\begin{aligned}
LHS - RHS &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-2}\frac{d}{dr}\left(r\frac{d\phi}{dr}(r) - \phi(r)\right) - r^{2k-1}\frac{d^2\phi}{dr^2}(r)\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\frac{d^2\phi}{dr^2}(r) - r^{2k-1}\frac{d^2\phi}{dr^2}(r)\right) \\
&= 0
\end{aligned}
$$

(ii) For $k = 1$, we have

$$
r\phi(r) = \beta_0^0 r\phi(r)
$$

By (iii), we have $\beta_0^0 = 1$. Now, assume that the result holds for $k-1$,

$$
\left(\frac{1}{r}\frac{d}{dr}\right)^{k-2}(r^{2k-3}\phi(r)) = \sum_{j=0}^{k-2}\beta_j^{k-1}r^{j+1}\frac{d^j\phi}{dr^j}
$$

then for k, we have

$$
LHS = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}\phi(r)) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-2}\left((2k-1)r^{2k-3}\phi(r) + r^{2k-2}\frac{d\phi}{dr}(r)\right)
$$

$$
RHS = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j} = \sum_{j=0}^{k-2}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j} + \beta_{k-1}^k r^k\frac{d^{k-1}\phi}{dr^{k-1}}
$$

By the induction hypothesis, the first term of RHS could be merged with the first term of LHS. Therefore, we have

$$
LHS - RHS = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-2} r^{2k-2}\frac{d\phi}{dr}(r) - \beta_{k-1}^k r^k\frac{d^{k-1}\phi}{dr^{k-1}}
$$

(iii) If we set $\phi(r) = 1$ and apply (ii), then we have the value of β_0^k for all k.

$\square$

Now we set

$$n = 2k + 1 \quad (k \geq 1).$$

If we suppose $u \in C^{k+1}(\mathbb{R}^n \times [0, \infty))$ solves the intial value problem (16). Then the function U defined by 17 is in $C^{k+1}(\mathbb{R}^n \times [0, \infty))$. Next, we introduce the new notations:

$$\begin{cases} \tilde{U}(r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}U(\mathbf{x};r,t)) \\ \tilde{G}(r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}G(\mathbf{x};r,t)) \\ \tilde{H}(r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}H(\mathbf{x};r,t)) \end{cases} \quad (r > 0, t \geq 0) \tag{34}$$

Then,

$$\tilde{U}(r,0) = \tilde{G}(r),\ \tilde{U}_t(r,0) = \tilde{H}(r) \tag{35}$$

We combine Lemma 2.1 and the identities provided by Lemma 2.2 to demonstrate the transformation (34) of U into $\tilde{U}$ in effect converts the Euler-Poisson-Darboux equation (16) into wave equation:

Lemma 2.3. *($\tilde{U}$ solves the one-dimesional wave equation) We have:*

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \quad \textit{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U}(r,0) = \tilde{G}(r),\ \tilde{U}_t(r,0) = \tilde{H}(r) \quad \textit{on } \mathbb{R}_+ \times \{t = 0\} \\ \tilde{U} = 0 \quad \textit{on } \{r = 0\} \times (0,\infty) \end{cases}$$

Proof. If $r > 0$, then by (i) of Lemma 2.2, we have

$$\begin{aligned} \tilde{U}_r r &= \left(\frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)(r^{2k-1}U) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}(r^{2k}U_r) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)(r^{2k}U_r) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}[r^{2k-1}U_{rr} + 2kr^{2k-2}U_r] \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\left(U_{rr} + \frac{n-1}{r}U_r\right)\right] \quad (n = 2k+1) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}(r^{2k-1}U_{tt}) \quad \text{by (18)} \\ &= \tilde{U}_{tt} \end{aligned}$$

It is clear that the next 3 equations holds according ro 18. By (ii) of Lemma 2.2, we have $\tilde{U} = 0$ on $\{r = 0\}$. Therefore, $\tilde{U}$ solves the one-dimensional wave equation. $\square$

Since $\tilde{U}$ is a solution of the on-demensional wave equation on the half line, we can apply the d'Alembert formula (15) to obtain the following representation of $\tilde{U}$:

$$\tilde{U}(r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{r-t}^{r+t} \tilde{H}(s)ds \tag{36}$$

for all $r \in \mathbb{R}, t > 0$. Recall:

$$u(\mathbf{x},t) = \lim_{r\to 0} U(\mathbf{x};r,t)$$

Futhermore, by (ii) in Lemma 2.2, we have

$$\begin{aligned}
\tilde{U}(r,t) &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}U(\mathbf{x};r,t)) \\
&= \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(\mathbf{x};r,t) \\
&= \beta_0^k r U(\mathbf{x};r,t) + \sum_{j=1}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(\mathbf{x};r,t)
\end{aligned}$$

Therefore,

$$\beta_0^k r U(\mathbf{x};r,t) = \tilde{U}(r,t) - \sum_{j=1}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(\mathbf{x};r,t)$$

which implies

$$U(\mathbf{x};r,t) = \frac{1}{\beta_0^k r}\tilde{U}(r,t) - \frac{1}{\beta_0^k r}\sum_{j=1}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} U(\mathbf{x};r,t)$$

Therefore, we have

$$u(\mathbf{x},t) = \lim_{r\to 0} U(\mathbf{x};r,t) = \lim_{r\to 0} \frac{1}{\beta_0^k r}\tilde{U}(r,t).$$

Thus, (36) implies

$$\begin{aligned}
u(\mathbf{x},t) &= \lim_{r\to 0} \frac{1}{\beta_0^k r}\left[\frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{r-t}^{r+t} \tilde{H}(s)ds\right] \\
&= \lim_{r\to 0} \frac{1}{\beta_0^k}\left[\left(\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r}\right) + \frac{1}{2r}\int_{r-t}^{r+t} \frac{\tilde{H}(s)}{r} ds\right] \\
&= \frac{1}{\beta_0^k}[\tilde{G}'(t) + \tilde{H}(t)]
\end{aligned}$$

We recall that

$$\tilde{G}(\mathbf{x},r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}G(\mathbf{x};r))$$

Now since $n = 2k+1$, it implies that $k = \frac{n-1}{2}$. Therefore, we have

$$\tilde{G}(\mathbf{x},t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2}G(\mathbf{x};r)\right)$$

By the definition of $G(\mathbf{x};r)$, we have

$$\tilde{G}(\mathbf{x},t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2}\fint_{\partial B(\mathbf{x},t)} g(\mathbf{y})dS(\mathbf{y})\right)$$

Similarly,

$$\tilde{H}(\mathbf{x},t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2}\fint_{\partial B(\mathbf{x},t)} h(\mathbf{y})dS(\mathbf{y})\right)$$

Therefore, we have this representation of $u(\mathbf{x},t)$:

$$\begin{cases} u(\mathbf{x},t) & = \frac{1}{\gamma_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(\mathbf{x},t)} g(\mathbf{y})dS(\mathbf{y})\right) \\ & +\frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(\mathbf{x},t)} h(\mathbf{y})dS(\mathbf{y})\right) \\ & \text{where } n \text{ is odd and } \gamma_n = 1\cdot 3\cdot 5\cdot\dots\cdot(n-2) \end{cases} \tag{37}$$

for $x \in \mathbb{R}^n, t > 0$. We notice that $\gamma_3 = 1$, so the representation of $u(\mathbf{x},t)$ in (37) agrees with $n = 3$ with (27). We still need to check the formula (37) really gives us a solution of (4).

Theorem 2.2. *(Solution of wave equation in odd dimensions) Assume now n is an odd integer, $n \geq 3$, and suppose also $g \in C^{m+1}\mathbb{R}^n, h \in C^m(\mathbb{R}^n)$, for $m = \frac{n+1}{2}$. Define $u(\mathbf{x},t)$ by (37). Then*

(i) $u \in C^2(\mathbb{R}^n \times [0,\infty))$,

(ii) $u_{tt} - \Delta u = 0$ *in* $\mathbb{R}^n \times (0,\infty)$,

(iii) $\lim_{(\mathbf{x},t)\to(\mathbf{x}^0,\mathbf{0}^+)} = g(\mathbf{x}^0)$, $\lim_{(\mathbf{x},t)\to(\mathbf{x}^0,\mathbf{0}^+)} = h(\mathbf{x}^0)$ *for each point* $\mathbf{x}^0 \in \mathbb{R}^n$.

Proof. 1. We suppose $g \equiv 0$, so

$$u(\mathbf{x},t) = \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2}H(\mathbf{x};t)\right) \tag{38}$$

By (i) in Lemma 2.2, we could compute $u_t t$:

$$u_{tt} = \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \left(t^{n-1}H_t(\mathbf{x};t)\right) \tag{39}$$

We use the same trick as before,

$$H_t(\mathbf{x};t) = \frac{t}{n}\fint_{B(\mathbf{x},t)} \Delta h(\mathbf{y})d\mathbf{y}$$

Therefore, by the definition of average ball integral, we have

$$
\begin{aligned}
u_{tt} &= \frac{1}{n\alpha(n)\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}}\left(\int_{B(\mathbf{x},t)} \Delta h(\mathbf{y})d\mathbf{y}\right) \\
&= \frac{1}{n\alpha(n)\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(\frac{1}{t}\int_{\partial B(\mathbf{x},t)} \Delta h(\mathbf{y})dS(\mathbf{y})\right)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Delta u(\mathbf{x},t) &= \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}(t^{n-2}\Delta H(\mathbf{x}:t)) \\
&= \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left[t^{n-2}\Delta_{\mathbf{x}}\left(⨍_{\partial B(\mathbf{x},t)} h(\mathbf{y})dS(\mathbf{y})\right)\right] \\
&= \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left[t^{n-2}⨍_{\partial B(\mathbf{x},t)} \Delta h(\mathbf{y})dS(\mathbf{y})\right]
\end{aligned}
$$

Then, by the definition of average ball integral, we have

$$
\Delta u = \frac{1}{n\alpha(n)\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(\frac{1}{t}\int_{\partial B(\mathbf{x},t)} \Delta h(\mathbf{y})dS(\mathbf{y})\right) = u_{tt}
$$

A similar calculation can be done when $h \equiv 0$.

2. If we choose the correct intial conditions g and h, then we can show that u is a solution of (4).

$\square$

Remark 2.3. *(i) Observing the formula, we need only have information of g, h and their derivatives on the sphere $\partial B(\mathbf{x}, t)$, not in the whole ball $B(\mathbf{x}, t)$.*

(ii) Comparing the formula (37) *with* (13)*, we notice that d'Alembert's formula does not the the derivative of g. This suggests that for $n > 1$, a solution of the wave equation neets not to be as smooth as the initial value g.*

2.3.4 Solution for even n

Assume now

$$n \text{ is an even integer, } n \geq 4,$$

Suppose u is a C^m solution of (4) in $\mathbb{R}^n \times (0,\infty)$, where $m = \frac{n+2}{2}$. The trick is the similar as the case when $n = 2$, which is called the method of descent. We define

$$\overline{u}(x_1, \cdots, x_n, x_{n+1}, t) := u(x_1, \cdots, x_n, t) \tag{40}$$

solves the wave equation in $\mathbb{R}^{n+1} \times (0,\infty)$, with initial conditions

$$\overline{u} = \overline{g}, \overline{u}_t = \overline{h} \quad \text{ on } \mathbb{R}^{n+1} \times \{t = 0\}$$

where

$$\begin{cases} \overline{g}(x_1, \cdots, x_n, x_{n+1}) := g(x_1, \cdots, x_n) \\ \overline{h}(x_1, \cdots, x_n, x_{n+1}) := h(x_1, \cdots, x_n) \end{cases} \tag{41}$$

Since $n+1$ is odd, we may apply (37)(with $n+1$ to replace n) to $\overline{u}$ to obtain a representation formula for $\overline{u}$ in terms of $\overline{g}, \overline{h}$. To carry out the details, let us fix $\mathbf{x} \in \mathbb{R}^n$, $t > 0$, and write $\overline{\mathbf{x}} = (\mathbf{x}, 0)$i.e. $\overline{\mathbf{x}} = (x_1, \cdots, x_n, 0) \in \mathbb{R}^{n+1}$. Then (37) gives with $n+1$ to replace n:

$$\begin{aligned} u(\mathbf{x}, t) = \frac{1}{\gamma_{n+1}} \Bigg[& \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^{n-1} \fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{g} d\bar{S} \right) \\ & + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^{n-1} \fint_{\partial \bar{B}(\bar{\mathbf{x}}, t)} \bar{h} d\bar{S} \right) \Bigg] \end{aligned}$$

where $\gamma_{n+1} = 1 \cdot 3 \cdots (n-1)$ and $B(\mathbf{x}, t)$ denoting the ball in $\mathbb{R}^{n+1}$ with center $\mathbf{x}$ and radius t, and $d\overline{S}$ denoting the n-dimensional surface measure on $\partial B(\overline{\mathbf{x}}, t)$. Now, we observe that

$$\fint_{\partial B(\overline{\mathbf{x}}, t)} \overline{g}(\overline{\mathbf{y}}) dS(\overline{\mathbf{y}}) = \frac{1}{(n+1)\alpha(n+1)t^n} \int_{\partial \overline{B}(\overline{\mathbf{x}}, t)} \overline{g}(\overline{\mathbf{y}}) dS(\overline{\mathbf{y}}) \tag{42}$$

Notice that $\partial \overline{B}(\overline{\mathbf{x}}, t) \cap \{y_{n+1} \geq 0\}$ is the graph of the function $\gamma(\mathbf{y}) = (t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}$. Similarly, $\partial \overline{B}(\overline{\mathbf{x}}, t) \cap \{y_{n+1} \leq 0\}$ is the graph of the function $-\gamma(\mathbf{y})$. Thus, (42) implies:

$$\fint_{\partial B(\overline{\mathbf{x}}, t)} \overline{g}(\overline{\mathbf{y}}) dS(\overline{\mathbf{y}}) = \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(\mathbf{x}, t)} g(\mathbf{y})(1 + |D\gamma(\mathbf{y})|^2)^{\frac{1}{2}} d\mathbf{y} \tag{43}$$

There is a "2" in the denominator because $\partial \overline{B}(\overline{\mathbf{x}}, t)$ consists of two hemisphere. Now,

$$(1 + |D\gamma(\mathbf{y})|^2)^{\frac{1}{2}} = \frac{t}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}}$$

We substitute this into (43) to obtain

$$\begin{aligned} \fint_{\partial B(\overline{\mathbf{x}}, t)} \overline{g}(\overline{\mathbf{y}}) dS(\overline{\mathbf{y}}) &= \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})t}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y} \\ &= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)} \fint_{B(\mathbf{x}, t)} \frac{g(\mathbf{y})}{(t^2 - |\mathbf{y} - \mathbf{x}|^2)^{\frac{1}{2}}} d\mathbf{y} \end{aligned}$$

Similarly, for h, we have

$$\fint_{\partial B(\overline{\mathbf{x}},t)} \overline{h}(\overline{\mathbf{y}})dS(\overline{\mathbf{y}}) = \frac{2t\alpha(n)}{(n+1)\alpha(n+1)}\fint_{B(\mathbf{x},t)} \frac{h(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{\frac{1}{2}}}d\mathbf{y}$$

We substitute these into the representation formula for u to obtain

$$u(\mathbf{x},t) =$$
$$\frac{1}{\gamma_{n+1}}\frac{2\alpha(n)}{(n+1)\alpha(n+1)}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\fint_{B(\mathbf{x},t)}\frac{g(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{1/2}}d\mathbf{y}\right)\right.$$
$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n\fint_{B(\mathbf{x},t)}\frac{h(\mathbf{y})}{(t^2-|\mathbf{y}-\mathbf{x}|^2)^{1/2}}d\mathbf{y}\right)\right].$$

Since $\gamma_{n+1} = 1\cdot 3\cdots(n-1)$ and

$$\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

where Γ is the Gamma function,

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$$

Therefore,

$$\begin{aligned}\frac{1}{\gamma_{n+1}}\frac{2\alpha(n)}{(n+1)\alpha(n+1)} &= \frac{1}{1\cdot 3\cdots(n-1)}\frac{2\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})}}{(n+1)\frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})}}\\ &= \frac{1}{1\cdot 3\cdots(n+1)}\frac{1}{\pi^{\frac{1}{2}}}\frac{\Gamma(\frac{n+3}{2})}{\Gamma(\frac{n+2}{2})}\end{aligned}$$

Using the property of Gamma function,

$$\Gamma(m+1) = m\Gamma(m)$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

We could conclude that

$$\Gamma\left(\frac{n+3}{2}\right) = \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

and

$$\Gamma\left(\frac{n+2}{2}\right) = \left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)\cdots\left(\frac{2}{2}\right)$$

Therefore,

$$\frac{1}{\gamma_{n+1}} \frac{2\alpha(n)}{(n+1)\alpha(n+1)} = \frac{1}{2 \cdot 4 \cdots (n-2) \cdot n}$$

We substitute this into the representation formula for u to obtain the fomula for even dimensions:

$$\begin{cases} u(\mathbf{y}, t) = \dfrac{1}{\gamma_n} \left[\left(\dfrac{\partial}{\partial t} \right) \left(\dfrac{1}{t} \dfrac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^n \fint_{B(\mathbf{x},t)} \dfrac{g(\mathbf{y})}{(t^2 - |\mathbf{y}-\mathbf{x}|^2)^{1/2}} d\mathbf{y} \right) \right. \\ \left. + \left(\dfrac{1}{t} \dfrac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left(t^n \fint_{B(\mathbf{x},t)} \dfrac{h(\mathbf{y})}{(t^2 - |\mathbf{y}-\mathbf{x}|^2)^{1/2}} d\mathbf{x} \right) \right] \end{cases} \tag{44}$$

where $\gamma_n = 2 \cdot 4 \cdots (n-2) \cdot n$ for $\mathbf{x} \in \mathbb{R}^n, t > 0$ and even $n \geq 2$. Since $\gamma_2 = 2$, it agress with Poisson's formula (33) if $n = 2$. Hence, we got the following theorem:

Theorem 2.3. *(Solution of wave equation in even dimensions) Assume n is an even integer, $n \geq 2$, and suppose also $g \in C^{m+1}(\mathbb{R}^n), h \in C^m(\mathbb{R}^n)$, for $m = \frac{n+2}{2}$. Define u by (38). Then*

(i) $u \in C^2(\mathbb{R}^n \times [0, \infty))$,

(ii) $u_{tt} - \Delta u = 0 \quad$ *in* $\mathbb{R}^n \times (0, \infty)$,

(iii) $\lim_{\substack{(\mathbf{x},t) \to (\mathbf{x}^0, 0) \\ \mathbf{x} \in \mathbb{R}^n, t > 0}} u(\mathbf{x}, t) = g(\mathbf{x}^0), \lim_{\substack{(\mathbf{x},t) \to (\mathbf{x}^0, 0) \\ \mathbf{x} \in \mathbb{R}^n, t > 0}} u_t(\mathbf{x}, t) = h(\mathbf{x}^0)$

This follows from the Theorem 2.2.

Remark 2.4. *(i) To compute $u(\mathbf{x}, t)$ for even n, we need information on $u = g, u_t = h$ on all of $B(\mathbf{x}, t)$, and not just on $\partial B(\mathbf{x}, t)$.*

(ii) ***Huggen's principle**: Comparing* (37) *and* (44)*, we observe that if n is odd and $n \geq 3$, then the intial conditions g, h at a given point $\mathbf{x} \in \mathbb{R}^n$ affect the solution u only on the boundary $\{(\mathbf{y}, t) \mid t > 0, |\mathbf{x}-\mathbf{y}| = t\}$ of the cone $C(\mathbf{x}) = \{(\mathbf{y}, t) \mid t > 0, |\mathbf{x}-\mathbf{y}| < t\}$, On the other hand, if n is even the initial condtion g, h affect the solution u on the whole cone $C(\mathbf{x})$.*

3 References

Evans L. C. Partial Differential Equations[M]. American Mathematical Soc., 1998, 67-82